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AR ASSOCIATED WITH ANR-SEQUENCE AND SHAPE

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For a given ANR-sequence $(\boldsymbol{X}, \boldsymbol{A})$ associated with a pair (X, A) of compacta, a pair $(N(\boldsymbol{X}), N(\boldsymbol{A}))$ of compact AR's containing (X, A) as an unstable pair is constructed. The weak proper homotopy type of the pair $(N(\boldsymbol{X})-X, N(\boldsymbol{A})-A)$ determines the shape of (X, A) in the sense of Mardešić and Segal. Several applications of this result are given. A cohomological version of the Whitehead theorem in shape theory is proved.

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shape	absolute retract
proper map	weak proper homotopy type

1. Introduction

Let (X, A) be a pair of compacta. In the papers [10] and [11], for a given ANR-sequence $(\boldsymbol{X}, \boldsymbol{A})$ associated with (X, A), a pair $(M(\boldsymbol{X}), M(\boldsymbol{A}))$ of AR's in which (X, A) is nicely embedded was constructed and it was applied to investigate the shape property of (X, A). In this paper we shall give further applications.

We let $\mathcal{M}$ denote the shape category for pairs of compact metrizable spaces in the sense of Mardešić and Segal [17]. We define a category $\mathcal{R}$ as follows. Objects of $\mathcal{R}$ are pairs of compacta. For objects (X, A) and (Y, B) of $\mathcal{R}$, let (M_X, M_A) and (N_Y, N_B) be any pairs of compact AR's containing (X, A) and (Y, B) as unstable pairs respectively. (All definitions are given in Section 2.) Morphisms of (X, A) to (Y, B) are certain equivalence classes of proper maps from $(M_X - X, M_A - A)$ to $(N_Y - Y, N_B - B)$ as defined in Section 4. The following theorems are proved.

Theorem 1. *There exists a category isomorphism $S: \mathcal{M} \to \mathcal{R}$ such that $S(X, A) = (X, A)$ for each object (X, A) of $\mathcal{M}$.*

Theorem 2. *Let (X, A) and (Y, B) be pairs of compacta. Then the followings are equivalent.*

(i) $\mathrm{Sh}_{MS}(X, A) = \mathrm{Sh}_{MS}(Y, B)$.

(ii) *If* $(M_X, M_A) \in m(X, A)$ *and* $(M_Y, M_B) \in m(Y, B)$, *then* $(M_X - X, M_A - A) \simeq_{wp} (M_Y - Y, M_B - B)$.

(iii) *There exist* $(M_X, M_A) \in m(X, A)$ *and* $(M_Y, M_b) \in m(Y, B)$ *such that* $(M_X - X, M_A - A) \simeq_{wp} (M_Y - Y, M_B - B)$.

(iv) *If* $(M_X, M_A) \in m(X, A)$ *and* $(M_Y, M_B) \in (Y, B)$, *then* $(M_X - X, M_A - A) \simeq_p (M_Y - Y, M_B - B)$.

(v) *There exist* $(M_X, M_A) \in m(X, A)$ *and* $(M_Y, M_B) \in m(Y, B)$ *such that* $(M_X - X, M_A - A \simeq_p (M_Y - Y, M_B - B)$.

Here Sh_{MS} means the shape in the sense of Mardešić and Segal. Theorem 1 strengthens Theorem 1 of Chapman [4] and Theorem 2 is a variant of Theorem 2 of Chapman [4] by proper homotopy type. Our proof is simple. Theorem 2 was proved by Siebenman [22] in the absolute case. In the above theorems it is known [14] that we cannot replace $\mathcal{M}$ by the shape category $\mathfrak{B}$ and Sh_{MS} by the shape Sh in the sense of Borsuk [1]. We will obtain a full subcategory $\mathfrak{B}'$ of $\mathfrak{B}$ in which Theorems 1 and 2 hold (Theorems 3 and 4). Finally we give a cohomological version of the Whitehead theorem in shape theory which strengthens Theorem 4 of Mardešić [15, (7.3)].

Theorem 5. *Let X and Y be approximately 1-connected compacta with finite fundamental dimension. Then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *if and only if there is a shape morphism $f: X \to Y$ such that the homomorphism $f^*: \check{H}^k(Y) \to \check{H}^k(X)$ induced by f is an isomorphism for each* $k \leq \max(\mathrm{Fd}(X), \mathrm{Fd}(Y))$, *where $\check{H}$ means the Čech cohomology with integral coefficients.*

Throughout the paper all spaces are metrizable and a pair of spaces is a pair of a space and a closed subset. All maps are continuous. AR and ANR mean those for metric spaces. We denote by J the set of positive integers and by I the unit interval.

2. Preliminaries

Let (X, A) and (Y, B) be pairs of compacta such that $X \subset Y$ and $A \subset B$. A pair (X, A) is said to be *unstable* in (Y, B) [21, p. 346] if $X \cap B = A$ and there is a homotopy

$$H: (Y, B) \times I \to (Y, B)$$

such that

$$H(y, 0) = y \quad \text{for each } y \in Y$$

and

$$H(y,t)\in Y-X \quad \text{for } y\in Y \quad \text{and} \quad 0<t\leq 1.$$

Recall that a map $f: X\to Y$ is *proper* if $f^{-1}(C)$ is compact for every compactum C of Y. A map $f:(X,A)\to(Y,B)$ is *proper* if $f: X\to Y$ is proper. Maps $f,g:(X,A)\to(Y,B)$ are *properly homotopic* (notation: $f\simeq_p g$) if there exists a proper map

$$H:(X,A)\times I\to(Y,B)$$

such that

$$H(x,0)=f(x) \quad \text{and} \quad H(x,1)=g(x) \quad \text{for } x\in X.$$

Pairs (X,A) and (Y,B) are said to be of the *same proper homotopy type* if there exist proper maps $f:(X,A)\to(Y,B)$ and $g:(Y,B)\to(X,A)$ such that $gf\simeq_p i_{(X,A)}$ and $fg\simeq_p i_{(Y,B)}$ where $i_{(X,A)}$ denotes the identity map on (X,A). In this case we write $(X,A)\simeq_p(Y,B)$. According to T. A. Chapman [4] proper maps $f,g:(X,A)\to(Y,B)$ are said to be *weakly proper homotopic* (notation: $f\simeq_{wp} g$) if for each compactum C of Y there exists a compactum D of X and a map $H:(X,A)\times I\to(Y,B)$ such that

$$H(x,0)=f(x), \quad H(x,1)=g(x) \quad \text{for } x\in X$$

and

$$H((X-D)\times I)\cap C=\emptyset.$$

If $f:(X,A)\to(Y,B)$ and $g:(Y,B)\to(X,A)$ are proper maps such that $gf\simeq_{wp} i_{(X,A)}$, then (Y,B) *weakly properly homotopically dominates* (X,A) (notation: $(X,A)\leq_{wp}(Y,B)$). If, additionally, $fg\simeq_{wp} i_{(Y,B)}$, then (X,A) and (Y,B) are of the same weak *proper homotopy type* and we write $(X,A)\simeq_{wp}(Y,B)$.

For pairs of compacta there were defined two kinds of shape by Borsuk [1] and Mardešić and Segal [17]. We denote by $\mathrm{Sh}(X,A)$ the shape of (X,A) in the sense of Borsuk and by $\mathrm{Sh}_{MS}(X,A)$ the shape in the sense of Mardešić and Segal. It is known by [2], [18] and [13] that Sh and Sh_{MS} are equivalent for compacta but not generally for pairs of compacta.

3. AR associated with ANR-sequence

Let (X,A) be a pair of compacta. According to Mardešić and Segal [17] an inverse sequence of pairs of compact ANR's

$$(\mathbf{X},\mathbf{A})=\{(X_k,A_k),\ \pi_k^{k+1},\ k\in J\cup\{0\}\}$$

whose limit $\underline{\lim}(\mathbf{X},\mathbf{A})=(X,A)$ is said to be an ANR-*sequence associated with* (X,A), where we may assume that X_0 is a singleton and π_0^1 is the constant map. We recall the following construction for ANR-sequences introduced by [10] and [11] which are made use of throughout this paper. Let $(\boldsymbol{X}, \boldsymbol{A}) = \{(X_k, A_k), \pi_k^{k+1}\}$ be an ANR-sequence associated with (X, A). For each $k \in J \cup \{0\}$, $M_k(\boldsymbol{X})$ denotes the mapping cylinder obtained by X_k, X_{k+1} and $\pi_k^{k+1}: X_{k+1} \to X_k$, that is, $M_k(\boldsymbol{X})$ is obtained by identifying points $(x, 1) \in X_{k+1} \times \{1\}$ and $\pi_k^{k+1}(x) \in X_k$ for $x \in X_{k+1}$ in a topological sum $X_{k+1} \times I \cup X_k$. since X_0 is a singleton, $M_0(\boldsymbol{X})$ is a cone over X_1. Consider a topological sum

$$T = \bigcup_{k=0}^{\infty} M_k(\boldsymbol{X}).$$

For each $k \in J$, by identifying each point of $X_k \times \{0\}$ in $M_{k-1}(\boldsymbol{X})$ and the corresponding point of X_k in $M_k(\boldsymbol{X})$ we obtain from T a locally compact metrizable space $T(\boldsymbol{X})$ which is said to be an *infinite* telescope associated with $\boldsymbol{X}$. Set $N(\boldsymbol{X}) = T(\boldsymbol{X}) \cup X$. Give $N(\boldsymbol{X})$* the following topology. $T(\boldsymbol{X})$ is open in $N(\boldsymbol{X})$ with its proper topology. Let $x \in X$. If $k \in J$, let V be an open neighborhood of $\pi_k(x)$ in X_k, where π_k is the projection of X into X_k. For $m > k$, consider an open set $(\pi_k^m)^{-1} V \times [0, 1)$ of $M_{m-1}(\boldsymbol{X})$ where $[0, 1) = \{t: 0 \leq t < 1\}$ and $\pi_k^m = \pi_k^{k+1} \cdots \pi_{m-1}^m$. The collection of the sets of the form

$$\pi_k^{-1}(V) \cup \bigcup_{m=k+1}^{\infty} (\pi_k^m)^{-1} V \times [0, 1),$$

where V ranges over open neighborhoods of $\pi_k(x)$ in X_k for $k \in J$, forms a neighborhood basis of X in $N(\boldsymbol{X})$. Obviously $N(\boldsymbol{X})$ is a compactum and contains X as a closed set (cf. [10, Theorem 1]). Similarly, for the inverse sequence $\boldsymbol{A} = \{A_k, \pi_k^{k+1} \mid A\}$ we construct mapping cylinders $M_k(\boldsymbol{A})$, $k = 0, 1, 2, \ldots$, an infinite telescope $T(\boldsymbol{A})$ and a compactum $N(\boldsymbol{A})$. Each of $M_k(\boldsymbol{A})$, $T(\boldsymbol{A})$ and $N(\boldsymbol{A})$ is considered as a closed set of $M_k(\boldsymbol{X})$, $T(\boldsymbol{X})$ and $N(\boldsymbol{X})$ respectively. Note that

$$\left.\begin{aligned} &M_k(\boldsymbol{A}) = M_k(\boldsymbol{X}) \cap N(\boldsymbol{A}), \quad k = 0, 1, 2, \ldots, \\ &T(\boldsymbol{A}) = T(\boldsymbol{X}) \cap N(\boldsymbol{A}), \\ &A = X \cap N(\boldsymbol{A}). \end{aligned}\right\} \tag{3.1}$$

For $n = 0, 1, 2, \ldots$, let

$$N_n(\boldsymbol{X}) = \bigcup_{k=0}^{n} M_k(\boldsymbol{X}) \quad \text{and} \quad N_n(\boldsymbol{A}) = \bigcup_{k=0}^{n} M_k(\boldsymbol{A}).$$

Define a map

$$\nu_t: (N(\boldsymbol{X}), N(\boldsymbol{A})) \to (N_{n-1}(\boldsymbol{X}), N_{n-1}(\boldsymbol{A}))$$

* The referee pointed out that $N(\boldsymbol{X})$ has also been constructed by J. Krasinkeiwicz, On a construction of ANR-sets, Fund. Math. 92 (1976), 95–112. The authors wish to thank the referee for his kind indication.

as follows:

$$\left.\begin{aligned}&\nu_n(x)=\pi_n(x)\quad \text{for } x\in X,\\ &\nu_n(x,t)=\pi_n^{k+1}(x)\quad \text{for } (x,t)\in M_k(X),\ k=n,n+1,\ldots,\\ &\nu_n(x,t)=(x,t)\quad \text{for } (x,t)\in M_k(X),\ k=0,1,\ldots,n-1.\end{aligned}\right\}\tag{3.2}$$

Here $N_{-1}(X)=N_{-1}(A)=M_{-1}(X)=X_0$ (a single point) and ν_0 is the constant map to X_0. The following lemmas are easily proved (cf. [11, Lemmas 1 and 2]).

Lemma 1. *For an* ANR-*sequence* (X, A) *associated with a pair* (X, A) *of compacta,* $(N(X), N(A))$ *is a pair of compact* AR's *containing* (X, A) *as an unstable pair.* $\{N(X)-N_k(X): k=0,1,2,\ldots\}$ *and* $\{N(A)-N_l(A): k=0,1,2,\ldots\}$ *form neighborhood bases of* X *and* A *in* $N(X)$ *and* $N(A)$ *respectively.*

Lemma 2. *For each* $n=0,1,2,\ldots$, *there exists a homotopy*

$$\xi_n:(N(X),N(A))\times I\to(N(X),N(A))$$

such that $\xi_n(x,0)=x$ *and* $\xi_n(x,1)=\nu_n(x)$ *for* $x\in N(X)$, $\xi_n(x,t)=x$ *for* $x\in N_{n-1}(X)$ *and* $t\in I$, *and* $\xi_n^{-1}(N_{n-1}(X)-X_n)=(N_{n-1}(X)-X_n)\times I$.

We call $(N(X), N(A))$ an AR *pair associated with* (X, A) and $(T(X), T(A))$ an *infinite telescope* (or simply a *telescope*) *associated with* (X, A).

Lemma 3. *Suppose that* (M_X, M_A) *and* (N_X, N_A) *are pairs of* AR's *containing* (X, A) *as an unstable pair. Then there exists a map* $\xi:(M_X, M_A)\to(N_X, N_A)$ *such that*

$$\xi\,|\,(X,A)=1_{(X,A)}\quad\textit{and}\quad \xi(M_X-X)\subset N_X-X.\tag{3.3}$$

If $\xi,\eta:(M_X, M_A)\to(N_X, N_A)$ *satisfy the condition* (3.3), *then there exists a homotopy* $H:(M_X, M_A)\times I\to(N_X, N_A)$ *such that*

$$H(y,0)=\xi(y)\quad\textit{and}\quad H(y,1)=\eta(y)\quad\textit{for } y\in M_X,\tag{3.4}$$

$$H(x,t)=x\quad\textit{for } x\in X\quad\textit{and}\quad t\in I,\tag{3.5}$$

$$H((M_X-X)\times I)\subset N_X-X.\tag{3.6}$$

Proof. We use the same argument as in Sher [21, Lemma (2.1)]. The existence of a map ξ satisfying (3.3) follows from the lemma of Sher. Suppose that ξ, η are maps satisfying (3.3). Since N_X and N_A are AR's, there exists a homotopy $H':(M_X, M_A)\times I\to(N_X, N_A)$ such that $H'(x,0)=\xi(x)$ and $H'(x,1)=\eta(x)$ for $x\in M_X$ and $H'(x,t)=x$ for $x\in X$ and $t\in I$. Let Φ be a homotopy from $(N_X, N_A)\times I$ to (N_X, N_A) such that $\Phi(x,0)=x$ for $x\in N_X$ and

$$\Phi((N_X, N_A)\times(0,1])\subset(N_X-X, N_A-A).$$

Take a map $\alpha: M_X \times I \to I$ such that

$$\alpha^{-1}(0) = M_X \times (\{0\} \cup \{1\}) \cup X \times I.$$

Define $H: (M_X, M_A) \times I \to (N_X, N_A)$ by

$$H(x, t) = \Phi(H'(x, t), \alpha(x, t)) \quad \text{for } (x, t) \in M_X \times I.$$

Obviously H satisfies (3.4), (3.5) and (3.6).

Lemma 4. *If $(\boldsymbol{X}, \boldsymbol{A})$ and $(\boldsymbol{X}', \boldsymbol{A}')$ are* ANR*-sequences associated with a pair (X, A) of compacta, then there exists a map $\xi: (N(\boldsymbol{X}), N(\boldsymbol{A})) \to (N(\boldsymbol{X}'), N(\boldsymbol{A}'))$ such that $\xi \mid (X, A) = 1_{(X,A)}$ and $\xi \mid (T(\boldsymbol{X}), T(\boldsymbol{A})): (T(\boldsymbol{X}), T(\boldsymbol{A})) \to (T(\boldsymbol{X}'), T(\boldsymbol{A}'))$ is a proper homotopy equivalence.*

The lemma follows from Lemmas 1 and 3.

4. Theorems

We let $\mathcal{M}$ denote the shape category for pairs of compacta defined by Mardešić and Segal [17]. Objects of $\mathcal{M}$ are pairs of compacta. Let us recall the definition of morphisms in $\mathcal{M}$ ([17, p. 42]). For ANR-sequences $(\boldsymbol{X}, \boldsymbol{A}) = \{(X_i, A_i), \pi_i^{i+1}\}$ and $(\boldsymbol{Y}, \boldsymbol{B}) = \{(Y_i, B_i), \mu_i^{i+1}\}$ a (system *map* $\boldsymbol{f}: (\boldsymbol{X}, \boldsymbol{A}) \to (\boldsymbol{Y}, \boldsymbol{B})$ consists of an increasing function $f: J \to J$ and a sequence $\{f_i: i \in J\}$ of maps $f_i: (X_{f(i)}, A_{f(i)}) \to (Y_i, B_i)$ such that for $i \leq j$ we have

$$f_i \pi_{f(i)}^{f(j)} \simeq \mu_i^j f_j: (X_{f(j)}, A_{f(j)}) \to (Y_i, B_i). \tag{4.1}$$

Two maps $\boldsymbol{f} = \{f, f_i\}, \boldsymbol{g} = \{g, g_i\}: (\boldsymbol{X}, \boldsymbol{A}) \to (\boldsymbol{Y}, \boldsymbol{B})$ are *homotopic* (notation: $\boldsymbol{f} \simeq \boldsymbol{g}$) if for each $i \in J$ there is $j \in J$ such that $j \geq f(i), g(i)$ and

$$f_i \pi_{f(i)}^j \simeq g_i \pi_{f(i)}^j : (X_j, A_j) \to (Y_i, B_i). \tag{4.2}$$

If $(\boldsymbol{X}, \boldsymbol{A})$, $(\boldsymbol{X}', \boldsymbol{A}')$ are both ANR-sequences associated with (X, A), there exists an identity system map $\boldsymbol{1}_{(\boldsymbol{X},\boldsymbol{A})(\boldsymbol{X}',\boldsymbol{A}')}$ (see [17, p. 46 and Theorem 10]). For pairs of compacta (X, A) and (Y, B), let $(\boldsymbol{X}, \boldsymbol{A})$, $(\boldsymbol{X}', \boldsymbol{A}')$, $(\boldsymbol{Y}, \boldsymbol{B})$ and $(\boldsymbol{Y}', \boldsymbol{B}')$ be ANR-sequences associated with (X, A) and (Y, B), respectively. System maps $\boldsymbol{f}: (\boldsymbol{X}, \boldsymbol{A}) \to (\boldsymbol{Y}, \boldsymbol{B})$ and $\boldsymbol{g}: (\boldsymbol{X}', \boldsymbol{A}') \to (\boldsymbol{Y}', \boldsymbol{B}')$ are said to be *equivalent* (notation: $\boldsymbol{f} \equiv \boldsymbol{g}$) if

$$\boldsymbol{1}_{(\boldsymbol{Y},\boldsymbol{B})(\boldsymbol{Y}',\boldsymbol{B}')} \cdot \boldsymbol{f} \simeq \boldsymbol{g} \cdot \boldsymbol{1}_{(\boldsymbol{X},\boldsymbol{A})(\boldsymbol{X}',\boldsymbol{A}')}$$

($\cdot$ means the composition [17, p. 42]). Morphisms in $\mathcal{M}$ are equivalence classes of system maps between ANR-sequences associated with pairs of spaces.

For a pair (X, A) of compacta, we denote by $m(X, A)$ the set of pairs (M_X, M_A) of compact AR's containing (X, A) as an unstable pair. For any pairs (M_X, M_A) and (N_X, N_A) in $m(X, A)$, choose a map

$$\xi'(M_X, N_X): (M_X, M_A) \to (N_X, N_A)$$

satisfying (3.3) of Lemma 3 and set

$$\xi(M_X, N_X) = \xi'(M_X, N_X) \mid (M_X - X, M_A - A).$$

By Lemma 3, $\xi(M_X, N_X)$ is a proper homotopy equivalence and its proper homotopy class is uniquely determined. For pairs of compacta (X, A) and (Y, B), let $(M_X, M_A), (N_X, N_A) \in m(X, A)$ and $(M_Y, M_B), (N_Y, N_B) \in m(Y, B)$. Maps $f: (M_X - X, M_A - A) \to (M_Y - Y, M_B - B)$ and $g: (N_X - X, N_A - A) \to (N_Y - Y, N_B - B)$ are said to be *equivalent* (notation: $f \equiv g$) if

$$\xi(M_Y, N_Y) \cdot f \simeq_{\mathrm{wp}} g \cdot \xi(M_X, N_X).$$

Let us define a category $\mathcal{R}$ as follows. Objects of $\mathcal{R}$ are pairs of compacta. For pairs of compacta (X, A) and (Y, B) morphisms of (X, A) into (Y, B) in $\mathcal{R}$ are equivalence classes of the collection of proper maps

$$\{f: (M_X - X, M_A - A) \to (M_Y - Y, M_B - B): \\ (M_X, M_A) \in m(X, A), (M_Y, M_B) \in m(Y, B)\}$$

under the equivalence relation $\equiv$. For pairs of compacta (X, A), (Y, B) and (Z, C), let $(M_X, M_A) \in m(X, A)$, (M_Y, M_B), $(N_Y, N_B) \in m(Y, B)$ and $(M_Z, M_C) \in m(Z, C)$. The composition of morphisms $\{f: (M_X - X, M_A - A) \to (M_Y - Y, M_B - B)\}$, $\{g: (N_Y - Y, N_B - B) \to (M_Z - Z, M_C - C)\}$ is the morphism $\{h: (M_X - X, M_A - A) \to (M_Z - Z, M_C - C)\}$ defined by

$$h = g \cdot \xi(M_Y, N_Y) \cdot f.$$

By Lemma 3 it is known that $\mathcal{R}$ forms a category.

Theorem 1. *There exists a category isomorphism $S: \mathcal{M} \to \mathcal{R}$ such that $S(X, A) = (X; A)$ for each object (X, A) of $\mathcal{M}$.*

For the proof we need a lemma. Let (X, A) and (Y, B) be pairs of compacta. Suppose $(\boldsymbol{X}, \boldsymbol{A}) = \{(X_i, A_i), \pi_i^{i+1}\}$ and $(\boldsymbol{Y}, \boldsymbol{B}) = \{(Y_i, B_i), \mu_i^{i+1}\}$ are ANR-sequences associated with (X, A) and (Y, B), respectively, and that $\boldsymbol{f} = \{f, f_i\}: (\boldsymbol{X}, \boldsymbol{A}) \to (\boldsymbol{Y}, \boldsymbol{B})$ is a map. Since $X_0 = Y_0 =$ a single point, we may assume that $f(0) = 0$ and $f_0: X_0 \to Y_0$ is the constant map. By (4.1), for each $k \in J$, there exists a homotopy

$$H_k: (X_{f(k+1)}, A_{f(k+1)}) \times I \to (Y_k, B_k)$$

such that

$$H_k(x, 0) = f_k \pi_{f(k)}^{f(k+1)}(x) \quad \text{and} \quad H_k(x, 1) = \mu_k^{k+1} f_{k+1}(x) \quad \text{for } x \in X_{f(k+1)}.$$

We define a map

$$\xi(\boldsymbol{f}): (T(\boldsymbol{X}), T(\boldsymbol{A})) \to (T(\boldsymbol{Y}), T(\boldsymbol{B}))$$

as follows. For each $k=0,1,2,\ldots$, consider the subsets $\bigcup_{i=f(k)}^{f(k+1)-1} M_i(X)$ and $M_k(Y)$ of $T(X)$ and $T(Y)$ respectively. Define a map

$$\xi_k: \bigcup_{i=f(k)}^{f(k+1)-1} M_i(X) \to M_k(Y)$$

by

$$\left.\begin{array}{l} \xi_k(x,t)=f_k\pi_{f(k)}^{j+1}(x) \quad \text{for } (x,t)\in M_j(X),\ j=f(k),\ldots,f(k+1)-2, \\ \xi_k(x,t)=\begin{cases}(f_{k+1}(x),2t) & \text{for } 0\le t\le 1/2,\ (x,t)\in M_{f(k+1)-1}(X),\\ H_k(x,2-2t) & \text{for } 1/2\le t\le 1,\ (x,t)\in M_{f(k+1)-1}(X).\end{cases}\end{array}\right\} \tag{4.3}$$

We define $\xi(f)$ by

$$\xi(f)\,\Big|\bigcup_{i=f(k)}^{f(k+1)-1} M_i(X)=\xi_k$$

for each $k=0,1,\ldots$. Obviously $\xi(f)$ is a proper continuous map of $(T(X), T(A))$ into $(T(Y), T(B))$. Although the construction $\xi(f)$ depends on the choice of the homotopies $\{H_k, k\in J\}$, the following lemma holds.

Lemma 5. *Let $[(X,A),(Y,B)]$ be the set of homotopy classes of maps from (X,A) into (Y,B) and $[(T(X),T(A)),(T(Y),T(B))]$ the set of weak homotopy classes of proper maps from $(T(X),T(A))$ into $(T(Y),T(B))$. Then ξ induces a $1:1$ correspondence $\tilde{\xi}$ from $[(X,A),(Y,B)]$ onto $[(T(X),T(A)),(T(Y),T(B))]$.*

Proof. Set $A=[(X,A),(Y,B)]$ and $B=[(T(X),T(A)),(T(Y),T(B))]$. Suppose that maps $f=\{f,f_i\}$, $g=\{g,g_i\}:(X,A)\to(Y,B)$ are homotopic. Let $\{L_k\}$ and $\{L_k'\}$ be sequences of homotopies used to construct proper maps $\xi(f)$ and $\xi(g)$, respectively. We have to prove $\xi(f)\simeq_w \xi(g)$. Let C be a compact set of $T(Y)$. By Lemma 1, there is $k\in J$ such that $C\subset$ Interior of $N_{k-1}(Y)$. Since $f\simeq g$, by (4.2), there exist $j\in J$, $j\ge f(k), g(k)$, and a homotopy $H_0:(X_j,A_j)\times I\to(Y_k,B_k)$ such that

$$H_0(x,0)=f_k\pi_{f(k)}^j(x) \quad\text{and}\quad H_0(x,1)=g_k\pi_{g(k)}^j(x) \quad \text{for } x\in X_j. \tag{4.4}$$

From the constructions of $\xi(f)$ and $\xi(g)$, we have

$$\xi(f)^{-1}(N_{k-1}(Y)-Y_k)\subset N_{f(k)-1}(X)-X_{f(k)}$$

and

$$\xi(g)^{-1}(N_{k-1}(Y)-Y_k)\subset N_{g(k)-1}(X)-X_{g(k)}.$$

Consider the maps

$$\xi(f)\nu_{f(k)},\ \xi(g)\nu_{g(k)}:(T(X),T(A))\to(T(Y),T(B)).$$

By Lemma 2, there exist homotopies $H_1,H_2:(T(X),T(A))\times I\to(T(X),T(B))$ such that $H_1(x,0)=\xi(f)(x)$, $H_1(x,1)=\xi(f)\nu_{f(k)}(x)$, $H_2(x,0)=\xi(g)(x)$, and

$H_2(x, 1) = \xi(g)\nu_{g(k)}(x)$ for $x \in T(X)$ and

$$\begin{aligned} &H_1^{-1}(C) \subset \{N_{f(k)-1}(X) - X_{f(k)}\} \times I \subset \{N_{j-1}(X) - X_j\} \times I, \\ &H_2^{-1}(C) \subset \{N_{g(k)-1}(X) - X_{g(k)}\} \times I \subset \{N_{j-1}(X) - X_j\} \times I. \end{aligned} \tag{4.5}$$

Note that

$$\xi(f)\nu_{f(k)}(x) = f_k \pi^j_{f(k)} \nu_j(x) \quad \text{and} \quad \xi(g)\nu_{g(k)}(x) = g_k \pi^j_{g(k)} \nu_j(x)$$

for each $x \in \{T(X) - N_{j-1}(B)\} \cup X_j$. Since $T(X)$ and $T(Y)$ are AR's, there exists a homotopy

$$H_3 : (T(X), T(A)) \times I \to (T(Y), T(B))$$

such that

$$H_3(x, 0) = \xi(f)\nu_{f(k)}(x) \quad \text{and} \quad H_3(x, 1) = \xi(g)\nu_{g(k)}(x) \quad \text{for } x \in T(X),$$

$$H_3(x, t) = H_0(\nu_j(x), t) \quad \text{for } x \in \{T(X) - N_{j-1}(X)\} \cup X_j \quad \text{and} \quad t \in I$$

(see (4.4)), and

$$H_3^{-1}(C) \subset (N_{j-1}(X) - X_j) \times I. \tag{4.6}$$

By connecting homotopies H_1, H_2 and H_3, we have a homotopy $H : (T(X), T(A)) \times I \to (T(Y), T(B))$ such that $H(x, 0) = \xi(f)(x)$, $H(x, 1) = \xi(g)(x)$ for $x \in T(X)$ and

$$H^{-1}(C) \cap ((T(X) - N_{j-1}(X)) \times I) = \emptyset$$

(cf. (4.5) and (4.6)). Thus $\xi(f) \simeq_{wp} \xi(g)$. Thus ξ induces a correspondence $\tilde{\xi} : A \to B$.

Next, suppose that a proper map $F : (T(X), T(A)) \to (T(Y), T(B))$ is given. We define a map $\eta(F)$ of (X, A) into (Y, B) as follows. For each $i \in J$, choose $f(i) \in J$ such that $F^{-1}(N_{i-1}(Y)) \subset N_{f(i)-1}(X)$ and $f(i) < f(j)$ if $i < j$. Since $F^{-1}(N_{i-1}(Y))$ is compact, from Lemma 1 such numbers exist. Define $f_i : (X_{f(i)}, A_{f(i)}) \to (Y_i, B_i)$ by $f_i = \nu_i F \mid X_{f(i)}$. It is easy to show that for $i < j$

$$\mu^j_i f_j \simeq f_i \pi^{f(j)}_{f(i)} : (X_{f(j)}, A_{f(j)}) \to (Y_i, B_i).$$

Thus $\eta(F) = \{f, f_i\}$ is a map of (X, A) into (Y, B). It is likewise easy to show that the homotopy class of $\eta(F)$ is independent of the choice of f. Suppose that $F, G : (T(X), T(A)) \to (T(Y), T(B))$ are proper maps such that $F \simeq_{wp} G$. To prove that $\eta(F) \simeq \eta(G)$, let $m \in J$. There exists a homotopy

$$H : (T(X), T(A)) \times I \to (T(Y), T(B))$$

and a compact set D of $T(X)$ such that

$$H((T(X) - D) \times I) \cap N_{m-1}(Y) = \emptyset.$$

Let $\eta(F) = \{f, f_i\}$ and $\eta(G) = \{g, g_i\}$. Choose $n \in J$ such that $n > f(m), g(m)$ and $D \subset$ Interior of $N_{n-1}(X)$. Then it is easy to show that $f_m \pi^n_{f(m)} \simeq g_m \pi^n_{g(m)}$. Thus $\eta(F) \simeq \eta(G)$. Therefore η induces a correspondence $\tilde{\eta} : B \to A$.

Finally, to complete the proof, it is enough to see that the equalities $\tilde{\eta}\tilde{\xi}=1_A$ and $\tilde{\xi}\tilde{\eta}=1_B$ hold. These equalities are easily proved by using the same argument as above and the definitions of $\tilde{\xi}$ and $\tilde{\eta}$.

Proof of Theorem 1. Suppose that (X, A) and (Y, B) are objects of $\mathcal{M}$ and $\varphi:(X, A)\to(Y, B)$ is a morphism in $\mathcal{M}$. Let $f:(\mathbf{X}, \mathbf{A})\to(\mathbf{Y}, \mathbf{B})$ be a map which represents φ, where $(\mathbf{X}, \mathbf{A})$ and $(\mathbf{Y}, \mathbf{B})$ are ANR-sequences associated with (X, A) and (Y, B) respectively. If $(N(\mathbf{X}), N(\mathbf{A}))$ and $(N(\mathbf{Y}), N(\mathbf{B}))$ are pairs of AR's associated with (X, A) and (Y, B), then $(N(\mathbf{X}), N(\mathbf{A}))\in m(X, A)$ and $(N(\mathbf{Y}), N(\mathbf{B}))\in m(Y, B)$ by Lemma 1. We define $S(\varphi)$ as the equivalence class containing $\xi(f)$, where $\xi(f)$ is a proper map defined in Lemma 5. That S is a functor of $\mathcal{M}$ to $\mathcal{R}$ and an isomorphism is easily proved by Lemma 5.

The following theorem has been proved by Siebenmann [22] in the absolute case. For completeness we give the proof.

Theorem 2. *Let (X, A) and (Y, B) be pairs of compacta. Then the following are equivalent.*

(i) $\mathrm{Sh}_{\mathrm{MS}}(X, A)=\mathrm{Sh}_{\mathrm{MS}}(Y, B)$.

(ii) *If $(M_X, M_A)\in m(X, A)$ and $(M_Y, M_B)\in m(Y, B)$, then $(M_X-X,\ M_A-A)\simeq_{wp}(M_Y-Y, M_B-B)$.*

(iii) *There exist $(M_X, M_A)\in m(X, A)$ and $(M_Y, M_B)\in m(Y, B)$ such that $(M_X-X, M_A-A)\simeq_{wp}(M_Y-Y, M_B-B)$.*

(iv) *If $(M_X, M_A)\in m(X, A)$ and $(M_Y, M_B)\in m(Y, B)$, then $(M_X-X, M_A-A)\simeq_p(M_Y-Y, M_B-B)$.*

(v) *There exist $(M_X, M_A)\in m(X, A)$ and $(M_Y, M_B)\in m(Y, B)$ such that $(M_X-X, M_A-A)\simeq_p(M_Y-Y, M_B-B)$.*

The equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) and (iv)$\Leftrightarrow$(v) follows from Theorem 1 and Lemma 3. Since the implication (v)$\Rightarrow$(iii) is trivial, it remains to prove the implication (i)$\Rightarrow$(v). To do it, we need the following lemma.

Lemma 6. *Let $f\simeq g:(X, A)\to(Y, B)$ and let (M_f, N_f) and (M_g, N_g) be the pairs of mapping cylinders obtained by (Y, B), (X, A), f and g respectively. Then there exist maps $\Phi:(M_f; N_f, X, A, Y, B)\to(M_g; N_g, X, A, Y, B)$ and $\Psi:(M_g; N_g, X, A, Y, B)\to(M_f; N_f, X, A, Y, B)$ such that*

$$\Phi\,|\,(X, A)=\Psi\,|\,(X, A)=1_{(X,A)},\tag{4.7}$$

$$\Phi\,|\,(Y, B)=\Psi\,|\,(Y, B)=1_{(Y,B)},\tag{4.8}$$

and homotopies $\Theta:(M_f; N_f, X, A, Y, B)\times I\to(M_f; N_f, X, A, Y, B)$ and $\Sigma:(M_g;$

$N_g, X, A, Y, B)\times I\to(M_g; N_g, X, A, Y, B)$ *such that*

$$\begin{aligned}&\Theta(x,0)=x \quad \text{and} \quad \Theta(x,1)=\Psi\Phi(x) \quad \text{for } x\in M_f,\\ &\Theta(x,s)=x \quad \text{for } x\in X\cup Y \quad \text{and} \quad s\in I,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&\Sigma(x,0)=x \quad \text{and} \quad \Sigma(x,1)=\Phi\Psi(x) \quad \text{for } x\in M_g,\\ &\Sigma(x,s)=x \quad \text{for } x\in X\cup Y \quad \text{and} \quad s\in I.\end{aligned} \tag{4.10}$$

Proof. The lemma is essentially proved by Puppe [19, § 2]. We give homotopies in exact form. There exists a homotopy $\xi:(X,A)\times I\to(Y,B)$ such that $\xi(x,0)=f(x)$ and $\xi(x,1)=g(x)$ for $x\in X$. Maps Φ, Ψ and homotopies Θ and Σ are defined as follows:

$$\begin{cases}\Phi(x,t)=(x,2t) & \text{for } x\in X \text{ and } 0\le t\le\frac12,\\ \Phi(x,t)=\xi(x,2-2t) & \text{for } x\in X \text{ and } \frac12\le t\le 1,\\ \Phi(y)=y & \text{for } y\in Y,\end{cases}$$

$$\begin{cases}\Psi(x,t)=(x,2t) & \text{for } x\in X \text{ and } 0\le t\le\frac12,\\ \Psi(x,t)=\xi(x,2t-1) & \text{for } x\in X \text{ and } \frac12\le t\le 1,\\ \Psi(y)=y & \text{for } y\in Y,\end{cases}$$

$$\begin{cases}\Theta((x,t),s)=(x,4t/(4-3s)) & \text{for } x\in X \text{ and } 0\le t\le(4-3s)/4,\\ \Theta((x,t),s)=\xi(x,4t+3s-4) & \text{for } x\in X \text{ and } (4-3s)/4\le t\le(2-s)/2,\\ \Theta((x,t),s)=\xi(x,2-2t) & \text{for } x\in X \text{ and } (2-s)/2\le t\le 1,\\ \Theta(y,s)=y & \text{for } y\in Y,\end{cases}$$

$$\begin{cases}\Sigma((x,t),s)=(x,4t/(4-3s)) & \text{for } x\in X \text{ and } 0\le t\le(4-3s)/4,\\ \Sigma((x,t),s)=\xi(x,5-4t-3s) & \text{for } x\in X \text{ and } (4-3s)/4\le t\le(2-s)/2,\\ \Sigma((x,t),s)=\xi(x,2t-1) & \text{for } x\in X \text{ and } (2-s)/2\le t\le 1,\\ \Sigma(y,s)=y & \text{for } y\in Y.\end{cases}$$

It is easy to see that Φ, Ψ, Θ *and* Σ satisfy (4.8), (4.9) and (4.10).

Now, let us complete the proof of Theorem 2 by showing the implication (i)$\Rightarrow$(v). Suppose that $\mathrm{Sh_{MS}}(X,A)=\mathrm{Sh_{MS}}(Y,B)$. There exist ANR-sequences $(\boldsymbol{X},\boldsymbol{A})=\{(X_i,A_i),\pi_i^{i+1}\}$ and $(\boldsymbol{Y},\boldsymbol{B})=\{(Y_i,B_i),\mu_i^{i+1}\}$ associated with (X,A) and (Y,B), and maps $\boldsymbol{f}=\{j,f_i\}:(\boldsymbol{X},\boldsymbol{A})\to(\boldsymbol{Y},\boldsymbol{B})$ and $\boldsymbol{g}=\{g,g_i\}:(\boldsymbol{Y},\boldsymbol{B})\to(\boldsymbol{X},\boldsymbol{A})$ such that $\boldsymbol{gf}\simeq\mathbf{1}_{(\boldsymbol{X},\boldsymbol{A})}$ and $\boldsymbol{fg}\simeq\mathbf{1}_{(\boldsymbol{Y},\boldsymbol{B})}$. If necessary, by taking cofinal sequences inductively, without loss of generality we may assume that

$$f(i)=i+1 \quad \text{and} \quad g(i)=i \quad \text{for } i\in J_0=J\cup\{0\}, \tag{4.11}$$

$$\pi_i^{i+1}\simeq g_if_i:(X_{i+1},A_{i+1})\to(X_i,A_i) \quad \text{for } i\in J_0, \tag{4.12}$$

$$\mu_i^{i+1}\simeq f_ig_{i+1}:(Y_{i+1},B_{i+1})\to(Y_i,B_i) \quad \text{for } i\in J_0. \tag{4.13}$$

We consider three ANR-sequences $(Z, C)=\{(Z_i, C_i), \nu_i^{i+1}\}$, $(Z', C')=\{(Z'_i, C'_i), \nu_i'^{i+1}\}$ and $(Z'', C'')=\{(Z''_i, C''_i), \nu_i''^{i+1}\}$ as follows:

$$(Z_i, C_i)=(X_i, A_i) \quad \text{and} \quad \nu_i^{i+1}=g_i f_i \quad \text{for } i\in J_0;$$

$$(Z'_{2i-1}, C'_{2i-1})=(X_i, A_i), \qquad (Z'_{2i}, C'_{2i})=(Y_i, B_i),$$

$$\nu'^{2i}_{2i-1}=g_i \quad \text{and} \quad \nu'^{2i+1}_{2i}=f_i \quad \text{for } i\in J_0;$$

$$(Z''_i, C''_i)=(Y_i, B_i) \quad \text{and} \quad \nu''^{i+1}_i=f_i g_{i+1} \quad \text{for } i\in J_0.$$

For each $i\in J$, by (4.12) and Lemma 6, there exist maps $\Phi_i:(M_i(X), M_i(A))\to(M_i(Z), M_i(C))$ and $\Psi_i:(M_i(Z), M_i(C))\to(M_i(X), M_i(A))$, homotopies Θ_i: $(M_i(X), M_i(A))\times I\to(M_i(X), M_i(A))$ and $\Sigma_i:(M_i(Z), M_i(C))\times I\to(M_i(Z), M_i(C))$ satisfying (4.7), (4.8), (4.9) and (4.10) for $(X, A)=(X_{i+1}, A_{i+1})$, $(Y, B)=(X_i, A_i)$, $f=\pi_i^{i+1}$ and $g=g_i f_i$. The families $\{\Phi_i: i\in J_0\}$ and $\{\Psi_i: i\in J_0\}$ define proper maps

$$\Phi:(T(X), T(A))\to(T(Z), T(C))$$

and

$$\Psi:(T(Z), T(C))\to(T(X), T(A)).$$

By using the families $\{\Theta_i\}$ and $\{\Sigma_i\}$ of homotopies we can show that Φ has a proper homotopy inverse Ψ. Thus $(T(X), T(A))\simeq_p(T(Z), T(C))$. Similarly, by using (4.13) and Lemma 6, we have $(T(Y), T(B))\simeq_p(T(Z''), T(C''))$. Since ANR-sequences (Z, C) and (Z', C') have the same limit space, it follows from Lemma 4 that $(T(Z), T(C))\simeq_p(T(Z'), T(C'))$. Similarly $(T(Z'), T(C'))\simeq_p(T(Z''), T(C''))$. Thus we have

$$(T(X), T(A))\simeq_p(T(Y), T(B)).$$

This completes the proof of Theorem 2.

For a compactum X, we denote by $C(X)$ a *cone* over X and by $0(X)$ an *open cone* $C(X)-X$.

Corollary 1. *In order that a pair (X, A) of compacta have the same shape in the sense of Mardešić and Segal with a pair (K, L) of compact* ANR*'s it is necessary and sufficient that for some (equivalently, every) pair $(M_X, M_A)\in m(X, A)$, $(M_X-X, M_A-A)\simeq_p(0(K), 0(L))$.*

Since $(C(K), C(L))\in m(K, L)$, the corollary is a consequence of Theorem 2.

Corollary 2. *A compactum X is of trivial shape if and only if for some (equivalently, every) compact* AR *$M_X\in m(X)$ $M_X-X\simeq_p$ a half line $(=[0, \infty))$.*

The proof is obvious.

Corollary 3. *A compactum X is an* FANR *(see Borsuk* [1]*) if and only if there exists a polyhedron K such that for some (equivalently, every) compact* AR $M_X \in m(X) M_X - X \leqslant_{wp} 0(K)$.

This follows from Borsuk [1, Chap. VIII (1.4)] and Mardešić [14, Remark 1].

Corollary 4. *Let P be one of the following spaces: n-sphere S^n, real projective n-space RP^n and complex projective n-space CP^n. If* $\mathrm{Sh}(X) < \mathrm{Sh}(P)$ *then X is of trivial shape.*

In the case $P = S^n$, the corollary was proved by Borsuk and Holsztyński [3]. To prove the corollary we need the following lemma.

Lemma 7. *Let $\mathcal{P}$ be a class consisting of connected polyhedra. If X and Y are compacta with* $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$, *and Y is $\mathcal{P}$-like (see* [12, p. 146]*), then there is a $\mathcal{P}$-like compactum X' such that* $\mathrm{Sh}(X) = \mathrm{Sh}(X')$.

Proof. Take ANR-sequences $\boldsymbol{X} = \{X_i, \pi_i^{i+1}\}$ and $\boldsymbol{Y} = \{Y_i, \mu_i^{i+1}\}$ maps $\boldsymbol{f} = \{f, f_i\} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} = \{g, g_i\} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$f(i) = i + 1 \quad \text{and} \quad g(i) = i \quad \text{for } i \in J_0 = J \cup \{0\}, \tag{4.14}$$

$$g_i f_i \simeq \pi_i^{i+1} \quad \text{for } i \in J_0, \tag{4.15}$$

$$Y_i \in \mathcal{P} \quad \text{for } i \in J_0, \tag{4.16}$$

$$X_i \text{ is connected and } f_i \text{ and } g_i \text{ are onto for } i \in J_0 \tag{4.17}$$

(see [23, *Proposition* 2]).

Consider the following ANR-sequences $\boldsymbol{Z} = \{Z_i, \nu_i^{i+1}\}$, $\boldsymbol{Z}' = \{Z_i', \nu_i'^{i+1}\}$ and $\boldsymbol{Z}'' = \{Z_i'', \nu_i''^{i+1}\}$ as follows:

$$Z_i = X_i \quad \text{and} \quad \nu_i^{i+1} = g_i f_i \quad \text{for } i \in J_0;$$

$$Z_{2i-1}' = X_i,\ Z_{2i}' = Y_i,\ \nu_{2i-1}'^{2i} = g_i \quad \text{and} \quad \nu_{2i}'^{2i+1} = f_i \quad \text{for } i \in J_0;$$

$$Z_i'' = Y_i \quad \text{and} \quad \nu_i''^{i+1} = f_i g_{i+1} \quad \text{for } i \in J_0.$$

By the same way as in the proof of the implication (i)$\Rightarrow$(v) of Theorem 2, it is proved that

$$T(\boldsymbol{X}) \simeq_p T(\boldsymbol{Z}) \simeq_p T(\boldsymbol{Z}') \simeq_p T(\boldsymbol{Z}'').$$

Set $X' = \varprojlim \boldsymbol{Z}''$. From Theorem 1 and [12, Theorem 1*], it follows that X' is a $\mathcal{P}$-like compactum such that $\mathrm{Sh}(X) = \mathrm{Sh}(X')$.

Proof of Corollary 4. By Lemma 7 we may assume that X is P-like. Since P is movable, X is movable. The corollary follows from [16, Theorem 4] for $P = S^n$, from [20, Theorem 3] and [8, Theorem 3.2] for $P = RP^n$ and from [23, Theorem 7] for $P = CP^n$.

It is known by K. Borsuk [2] and S. Mardešić [14] that there exist pairs of compacta (X, A) and (Y, B) such that $\mathrm{Sh}(X, A) \neq \mathrm{Sh}(Y, B)$ but $\mathrm{Sh}_{\mathrm{MS}}(X, A) = \mathrm{Sh}_{\mathrm{MS}}(Y, B)$.

According to [11, § 4] a closed set A of a space X is said to be *neighborhood deformable* if there exist a neighborhood U of A in X and a homtopy $H: U \times I \to X$ such that

$$H(x, 0) = x \quad \text{for } x \in U \quad \text{and} \quad H(U \times \{1\} \cup A \times I) \subset A.$$

A pair (X, A) of compacta is said to be an *α-pair* if there exists a pair of AR's $(M_X, M_A) \in m(X, A)$ such that for some closed neighborhood F of M_A in M_X there exists a proper homotopy $H: (F - X, M_A - A) \times I \to (M_X - X, M_A - A)$ satisfying

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) \in M_A - A \quad \text{for } x \in F - X. \tag{4.18}$$

By Lemma 3, if (X, A) is an α-pair then it is proved that for every $(M_X, M_A) \in m(X, A)$ there exist a closed neighborhood F of M_A in M_X and a homotopy H satisfying (4.18). Also, it is known by the proof of [10, Lemma 4] that if A is neighborhood deformable in X then (X, A) is an α-pair. Thus the following theorems are consequences of [11, Theorem 4], Theorems 1 and 2.

Theorem 3. *Let $\mathscr{B}'$ be the full subcategory of the shape category in the sense of Borsuk defined as follows: $(X, A) \in 0(\mathscr{B}')$ if and only if there is an α-pair (Y, B) such that* $\mathrm{Sh}(X, A) = \mathrm{Sh}(Y, B)$, *where $0(\mathscr{B}')$ means the set of objects of $\mathscr{B}'$. similarly let R' be the full subcategory of $\mathscr{R}$ such that $0(\mathscr{R}') = 0(\mathscr{B}')$. Then there exists a category isomorphism $S': \mathscr{B}' \to \mathscr{R}'$.*

Theorem 4. *Let (X, A) and (Y, B) be objects in $\mathscr{B}'$. Then each of the conditions* (ii), (iii), (iv) *and* (v) *in Theorem 2 is equivalent to the relation* $\mathrm{Sh}(X, A) = \mathrm{Sh}(Y, B)$.

We given a simple example which shows that the restrictions to the category $\mathscr{B}'$ in Theorems 3 and 4 are necessary.

Example 1. Consider the following subsets of the plane:

$$X = \{(0, 0)\} \cup \{(1/n, 0) : n \in J\}, \qquad A = \{(0, 0)\},$$

$$M_X = \{(x, y) : 0 \leq x, y \leq 1\}, \qquad M_A = \{(0, y) : 0 \leq y \leq 1\},$$

$$Y = \{(x, 0) : 0 \leq x \leq 1\} \cup \{(1 + 1/n, 0) : n \in J\}, \qquad B = \{(0, 0)\},$$

$$M_Y = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}, \qquad M_B = \{(0, y) : 0 \leq y \leq 1\}.$$

Obviously $(M_X, M_A) \in m(X, A)$ and $(M_Y, M_B) \in m(Y, B)$. Since $(M_X - X, M_A - A) =_p (M_Y - Y, M_B - B)$ (the two pairs are homeomorphic), we have $\mathrm{Sh}_{\mathrm{MS}}(X, A) = \mathrm{Sh}_{\mathrm{MS}}(Y, B)$. However $\mathrm{Sh}(X, A)$ and $\mathrm{Sh}(Y, B)$ are incomparable, that is, $\mathrm{Sh}(X, A) \not\leq \mathrm{Sh}(Y, B)$ and $\mathrm{Sh}(X, A) \not\geq \mathrm{Sh}(Y, B)$. Note that (Y, B) is an α-pair but (X, A) is not.

The following example shows that we can not replace the weak proper homotopy category by the proper homotopy category inTheorem 1.

Example 2. Let $\tilde{S}$ be the dyadic solenoid. Then there is an inverse sequence $S = \{S_i, \pi_i^{i+1}\}$ such that $\varprojlim S = \tilde{S}$, each S_i is a 1-sphere and the degree of $\pi_i^{i+1} = 2$ for each i. Take two points a, b in $\tilde{S}$ such that a and b do not belong to the same pathwise-connected component of $\tilde{S}$. Let $a_i = \pi_i(a)$, $b_i = \pi_i(b)$ where $\pi_i: \tilde{S} \to S_i$ is the projection map. Let us define two proper maps $f_a, f_b: [0, \infty) \to T(S)$ as follows;

$$f_a(n) = a_n \quad \text{for } n \in J,$$

$$f_a(n-t) = (a_n, t) \in M_{n-1}(S) \quad \text{for } 0 \leq t \leq 1,$$

$$f_b(n) = b_n \quad \text{for } n \in J,$$

$$f_b(n-t) = (b_n, t) \in M_{n-1}(S) \quad \text{for } 0 \leq t \leq 1,$$

where $a_0 = b_0$ is the vertex of the cone $M_0(S)$. Now we can show that $f_a \simeq_{wp} f_b$, $f_a \not\simeq_p f_b$. It is easy to prove that the set of all shape morphisms from the one point space to $\tilde{S}$ has only one element. Hence by Theorem 1, we have that $f_a \simeq_{wp} f_b$. Next we suppose that $f_a \simeq_p f_b$. Then there is a proper map $H: [0, \infty) \times I \to T(S)$ such that $H(x, 0) = f_a(x)$, $H(x, 1) = f_b(x)$ for $x \in [0, \infty)$. Take a $k_1 \in J$. Since H is proper, there exists $k_2 \in J$ such that

$$H^{-1}(M_{k_1}(S)) \subset \text{Interior of } [0, k_2] \times I.$$

Let

$$\varphi_k: (I, \{0\}, \{1\}) \to (S_{k_1}, \{a_{k_1}\}, \{b_{k_1}\})$$

be maps for each $k \geq k_2$ such that $\varphi_k(t) = \nu_{k_1} H(k, t)$ for $t \in I$. it is easy to prove that

$$\varphi_k \simeq \varphi_{k_2} \operatorname{rel}(\{0\}, \{1\}) \quad \text{for each } k, k > k_2. \tag{*}$$

However, by simple computation of the homomorphism

$$\varphi_{k*}: H_1(I, \{0\} \cup \{1\}) \to H_1(S_{k_1}, \{a_{k_1}\} \cup \{b_{k_1}\}),$$

it is known that there exists $k_3 \geq k_2$ such that $\varphi_{k_3} \not\simeq \varphi_{k_2} \operatorname{rel}(\{0\} \cup \{1\})$ (cf. Mardešić and Segal [16]). This contradicts (*). Hence $f_a \not\simeq_p f_b$.

The final theorem is a cohomological version of Whitehead theorem in shape theory. Such a theorem was first given by Mardešić [15, Theorem 4, 7.3]. We strengthen this theorem of Mardešić. Our proof is different from Mardešić's.

A compactum X is said to be *approximatively* 1*-connected* if there is an embedding i of X into an AR M satisfying the condition: For every neighborhood V of $i(X)$ in M there is a neighborhood V_0 of $i(X)$ such that every map of a 1-sphere into V_0 is null homotopic in V. The *fundamental dimension* of a compactum X (denoted by $\mathrm{Fd}(X)$) is the minimum of the dimensions of compacta Y with $\mathrm{Sh}(X) \leq \mathrm{Sh}(Y)$. (See [1, p. 31].)

Theorem 5. *Let X and Y be approximatively 1-connected compacta with finite fundamental dimension. Then* $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ *if and only if there is a shape morphism $f: X \to Y$ such that the homomorphism $f^*: \check{H}^k(Y) \to \check{H}^k(X)$ induced by f is an isomorphism for each $k \leq \max(\mathrm{Fd}(X), \mathrm{Fd}(Y))$, where $\check{H}$ means the Čech cohomology with integral coefficients.*

Proof. It is obvious that the only if part holds without any hypothesis. Let us prove the if part. Without loss of generality we may assume by [9, Cor. 1] that $\mathrm{Fd}(X)=\dim X$ and $\mathrm{Fd}(Y)=\dim Y$. We set $m=\max(\dim X, \dim Y)$. Since X and Y are approximately 1-connected, there exist ANR-sequences $\boldsymbol{X}=\{X_i, \pi_i^{i+1}\}$ and $\boldsymbol{Y}=\{Y_i, \mu_i^{i+1}\}$ such that

$$\boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ are polyhedra with dimension } \leqq m \tag{4.19}$$

for each i, π_i^{i+1} is piecewise linear and the homomorphisms
$(\pi_i^{i+1})_*: \pi_1(X_{i+1}, x) \to \pi_1(X_i, \pi_i^{i+1}(x))$
and $(\mu_i^{i+1})_*: \pi_1(Y_{i+1}, y) \to \pi_1(Y_i, \mu_i^{i+1}(y))$
are trivial for $x \in X_{i+1}$ and $y \in Y_{i+1}$. (4.20)

Suppose that $f: \boldsymbol{X} \to \boldsymbol{Y}$ is a map whose induced homomorphism $f^*: \check{H}^k(Y) \to \check{H}^k(X)$ is an isomorphism for each $k \leq m$. Let $T(\boldsymbol{X})$ and $T(\boldsymbol{Y})$ be infinite telescopes associated with $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. Construct a proper map $\xi=\xi(f): T(\boldsymbol{X}) \to T(\boldsymbol{Y})$ as in the paragraph preceding Lemma 5. By (4.19) and (4.20) $T(\boldsymbol{X})$ and $T(\boldsymbol{Y})$ are locally finite polytopes with dimension $\leq m+1$. If we denote by $N(\boldsymbol{X})$ and $N(\boldsymbol{Y})$ compact AR's associated with $\boldsymbol{X}$ and $\boldsymbol{Y}$ (see 3), then we have isomorphisms

$$H^k(X) \approx H^{k+1}(N(\boldsymbol{X}), X) \approx H_c^{k+1}(T(\boldsymbol{X}))$$

and

$$H^k(Y) \approx H^{k+1}(N(\boldsymbol{Y}), Y) \approx H_c^{k+1}(T(\boldsymbol{Y}))$$

for each $k \leq m$, where H_c^* means the singular cohomology with compact supports. From the definition of the map ξ it follows that

$$\xi^*: H_c^k(T(\boldsymbol{Y})) \to H_c^k(T(\boldsymbol{X})) \quad \text{for each } k \leq m+1.$$

Since $m+1 \geq \max(\dim T(\boldsymbol{X}), \dim T(\boldsymbol{Y}))$, we have

$$\xi^*: H_c^k(T(\boldsymbol{Y})) \approx H_c^k(T(\boldsymbol{X})) \quad \text{for } k=0,1,2,\ldots. \tag{4.21}$$

Now, to conclude that ξ is a proper homotopy equivalence, we appeal to Theorem 4.1 of [7]. Let us check that the hypothesis of Theorem 4.1 of [7] is satisfied. To prove that ξ is properly 0-connected, consider the group $H_c^0(T(X))$ and $H_c^0(T(Y))$ (see [7, p. 7]). From the continuity of Čech cohomology [6] it follows that $H_c^0(T(X)) \approx \check{H}^0(X)$ and $H_c^0(T(Y)) \approx \check{H}^0(Y)$. Since $f^*: \check{H}^0(Y) \approx \check{H}^0(X)$, we have

$$\xi^*: H_c^0(T(Y)) \approx H_c^0(T(X)).$$

Thus ξ is properly 0-connected. Also, by (4.20), we can conclude that

$$\Delta(T(X), \{p\}; \pi_1, \text{nocov}) = \Delta(T(Y), \{p\}, \pi_1, \text{nocov}) = 0.$$

Finally, since $T(X)$ and $\widetilde{T(Y)}$ are contractible,

$$H_*(T(X)) = H_*(T(X)) = H_*(T(Y)) = H_*(T(Y)) = 0,$$

where $\widetilde{T(X)}$ and $\widetilde{T(Y)}$ are the universal covering spaces of $T(X)$ and $T(Y)$. Thus the hypothesis of Theorem 4.1 of [7] has been satisfied. Therefore ξ is a proper homotopy equivalence. By Theorem 1 or Theorem 2 we have $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. This completes the proof.

From Theorem 5 of Draper and Keesling [5], it is known that we cannot omit the condition $\max(\mathrm{Fd})(X), \mathrm{Fd}(Y)) < \infty$ in Theorem 5.

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